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Geometric quantisation of the MIC–Kepler problem

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Abstract. The geometric quantisation scheme is applied to the compact Kaehler orbit manifolds of the modified Kepler problem. Thus we obtain the quantisation of the magnetic charge and energy spectrum of the corresponding quantum problem. A new regularisation of the standard Kepler problem is presented.

The interplay between geometric quantisation and reduction of Hamiltonian systems in the sense of Marsden and Weinstein (1974) has been discussed recently by many authors. Thus Guillemin and Sternberg (1982) obtain compatibility of quantisation and reduction in the case when the symplectic manifold to be reduced is a compact Kaehler manifold. At the other extreme, the reduction-quantisation relationship within the category of cotangent bundles is treated, e.g., by Puta (1984) and Gotay (1986). On the other hand, concrete quantum spectra of classical Hamiltonian systems (the hydrogen atom, the harmonic oscillator, cf Czyz (1979)) were obtained when a cotangent bundle (i.e. a classical phase space) is reduced to a compact Kaehler manifold where geometric quantisation gives the strongest implications.

In the present paper, we give a further example of the above procedure. We apply geometric quantisation (via reduction) to the Hamiltonian system:

$$(T^*\dot{R}^3, \Omega_\mu, H_\mu) \tag{1}$$

where

$$T^*\dot{R}^3 = \{(q,p) \in R^3 \times R^3 \colon q \neq 0\}$$

$$\Omega_\mu = \mathrm{d}\theta + \sigma_\mu \qquad \theta = \Sigma p_j\, \mathrm{d}q_j \qquad \sigma_\mu = (-\mu/|q|^3)\varepsilon_{ijk} q_i\, \mathrm{d}q_j \wedge \mathrm{d}q_k \tag{2}$$

$$H_\mu = \tfrac{1}{2}|p|^2 + \tfrac{1}{2}\mu^2/|q|^2 - \alpha/|q| \qquad \alpha, \mu \in R,\ \alpha > 0. \tag{3}$$

The Hamiltonian system (1) describes the motion of a charged particle in the presence of a Dirac monopole field $B_\mu = -\mu q/|q|^3$, a Newtonian potential $\alpha/|q|$, and a centrifugal potential $\mu^2/2|q|^2$. Using a vector potential this system has been studied by McIntosh and Cisneros (1970) and here, following Iwai and Uwano (1986) whose work has influenced substantially the present study, we adopt the name MIC-Kepler problem. It is known that the energy level submanifolds $H_\mu^{-1}(E)$ consist only of closed orbits for $E < 0$. When $\mu = 0$ we have the standard Kepler problem, which has been quantised geometrically by Simms (1973) and Mladenov and Tsanov (1985) in higher dimensions.

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In this case, even for $E<0$, there are some non-closed orbits (i.e. there is no $U(1)$ action). To circumvent this difficulty use has to be made of some regularisation procedure (see Moser 1970, Kummer 1982, Cordani 1986, Vivarelli 1986). In the present paper we regularise in a different way (see the remark after the proposition). We denote by $\mathcal{O}_\mu(E)$ the orbit manifold of the flow of the Hamiltonian H on the level set $H_\mu^{-1}(E)$. The level sets described in statement (ii) of theorem 1 below consist of fixed points of the flow.

Theorem 1. Let $E<0$, $\lambda=\sqrt{-8E}$. Then
 (i) if $\lambda|\mu|<2\alpha$ then $\mathcal{O}_\mu(E)=H_\mu^{-1}(E)/U(1)\simeq P^1\times P^1$
 (ii) if $\lambda|\mu|=2\alpha$ then $\mathcal{O}_\mu(E)=H_\mu^{-1}(E)/U(1)\simeq P^1$
 (iii) if $\lambda|\mu|>2\alpha$ then $H_\mu^{-1}(E)=\phi$.

Moreover, the reduced symplectic form on $\mathcal{O}_\mu(E)$ is

$$\Omega_\mu(E)=\frac{2\pi(2a+\lambda\mu)}{\lambda}\omega_1+\frac{2\pi(2\alpha-\lambda\mu)}{\lambda}\omega_2 \tag{4}$$

where

$$\omega_j=\frac{\mathrm{i}}{2\pi}\frac{\mathrm{d}\zeta_j\wedge\mathrm{d}\bar{\zeta}_j}{(1+|\zeta_j|^2)^2}\qquad j=1,2 \tag{5}$$

for any pair of non-homogeneous coordinates (ζ_1,ζ_2) on $P^1\times P^1$.

Theorem 1 reduces the quantisation of the MIC-Kepler problem to the geometric quantisation of a compact Kaehler manifold $P^1\times P^1$ (P^1). Applying the geometric quantisation scheme to the orbit manifold $\mathcal{O}_\mu(E)$ amounts in quantum mechanical terms to the transition from the Schrödinger to the Heisenberg picture and leads to the next theorem.

Theorem 2. The spectrum of the MIC-Kepler problem (1) (α and μ fixed) consists of the energy levels

$$E_N=-\alpha^2/2N^2\qquad N=|\mu|+1,|\mu|+2,\dots \tag{6}$$

with multiplicities

$$m(E_N)=N^2-\mu^2. \tag{7}$$

Theorems 1 and 2 will be proved later. We remark that the above method of quantisation was initiated by Simms (1973). For a detailed exposition of geometric quantisation see Simms and Woodhouse (1976), Sniatycki (1980) and Tuynman (1985).

We start with the symplectic manifold

$$T^*\dot{R}^4=\{(x,y)\in R^4\times R^4\colon x\neq 0\}$$

with the standard symplectic form

$$\Omega=\sum \mathrm{d}y_j\wedge\mathrm{d}x_j\qquad j=1,2,3,4. \tag{8}$$

For an arbitrary choice of a positive constant λ, we introduce the complex coordinates (which are a slight modification of the coordinates used by Iwai and Uwano (1986))

$$\begin{aligned}
z_1&=\lambda x_1+y_2+\mathrm{i}(\lambda x_2-y_1)=\lambda(x_1+\mathrm{i}x_2)-\mathrm{i}(y_1+\mathrm{i}y_2)\\
z_2&=\lambda x_3+y_4+\mathrm{i}(\lambda x_4-y_3)=\lambda(x_3+\mathrm{i}x_4)-\mathrm{i}(y_3+\mathrm{i}y_4)\\
z_3&=\lambda x_1-y_2-\mathrm{i}(\lambda x_2+y_1)=\lambda(x_1-\mathrm{i}x_2)-\mathrm{i}(y_1-\mathrm{i}y_2)\\
z_4&=\lambda x_3-y_4-\mathrm{i}(\lambda x_4+y_3)=\lambda(x_3-\mathrm{i}x_4)-\mathrm{i}(y_3-\mathrm{i}y_4).
\end{aligned}$$

Thus $T^*\dot{R}^4 = C^4\backslash D$ where

$$D = \{z \in C^4 \colon z_1 = -\bar{z}_3, z_2 = -\bar{z}_4\}. \tag{9}$$

Obviously our form Ω is a multiple of the standard Kaehler form on C^4 and, more precisely,

$$\Omega = \frac{\mathrm{i}}{4\lambda} \mathrm{d}z \wedge \mathrm{d}\bar{z} = \frac{\mathrm{i}}{4\lambda} \sum \mathrm{d}z_j \wedge \mathrm{d}\bar{z}_j. \tag{10}$$

We introduce three Hamiltonian functions on $T^*\dot{R}^4$. First the Hamiltonian of the *conformal Kepler problem*

$$H = (|y|^2 - 8\alpha)/8|x|^2 \qquad \alpha \text{ fixed positive constant} \tag{11}$$

second the Hamiltonian of a harmonic oscillator

$$K = \tfrac{1}{2}(\lambda^2|x|^2 + |y|^2) = |z|^2 = \sum z_j\bar{z}_j \tag{12}$$

and third a moment Hamiltonian

$$M(x, y) = \tfrac{1}{2}(x_1y_2 - x_2y_1 + x_3y_4 - x_4y_3) = (|z_1|^2 + |z_2|^2 - |z_3|^2 - |z_4|^2)/8\lambda. \tag{13}$$

The following two lemmas are known but we include some hints to proofs for completeness.

Lemma 1. Let $E < 0$, $\lambda = \sqrt{-8E}$. Then

$$H^{-1}(E) = K^{-1}(4\alpha). \tag{14}$$

Moreover, the flows defined by the Hamiltonians H and K on the energy hypersurfaces (14) coincide up to a monotone change of parameter.

Proof. Obviously we have

$$4|x|^2(H + \lambda^2/8) = K - 4\alpha \tag{15}$$

which proves (14). Also, the corresponding Hamiltonian vector fields X_H, X_K, restricted on the level sets (14) satisfy

$$4|x|^2 X_H = X_K \tag{16}$$

which proves the lemma.

We remark that the Hamiltonians K and M as well the symplectic form Ω are well defined on the manifold

$$\dot{C}^4 = C^4\backslash\{0\} \supset T^*\dot{R}^4. \tag{17}$$

We denote by K_t, M_s, the flows of the Hamiltonian systems $(K, \Omega, \dot{C}^4)$, $(M, \Omega, \dot{C}^4)$ respectively.

Lemma 2. For any $z \in \dot{C}^4$, s, $t \in R$, we have

$$K_t z = (\mathrm{e}^{\mathrm{i}\lambda t}z_1, \mathrm{e}^{\mathrm{i}\lambda t}z_2, \mathrm{e}^{\mathrm{i}\lambda t}z_3, \mathrm{e}^{\mathrm{i}\lambda t}z_4) \tag{18}$$

$$M_s z = (\mathrm{e}^{\mathrm{i}s/2}z_1, \mathrm{e}^{\mathrm{i}s/2}z_2, \mathrm{e}^{-\mathrm{i}s/2}z_3, \mathrm{e}^{-\mathrm{i}s/2}z_4). \tag{19}$$

In particular, the flows of all three Hamiltonians H, K, M, commute where defined.

Proof. Equations (18) and (19) are obtained by straightforward computation. The last statement of the lemma is a consequence of (18), (19) and lemma 1.

By lemma 2, the flows K_t, M_s define a symplectic action of the torus $U(1)\times U(1)$ on the manifold $\dot{C}^4$. We denote by

$$J:\dot{C}^4\to u^*(1)\times u^*(1)\simeq R^2 \tag{20}$$

the moment map of this action. Explicitly we have

$$J(z)=(M(z),K(z)).$$

We note that the set D defined in formula (9) is invariant with respect to the $U(1)$ action defined by the Hamiltonian M (19). Thus $T^*\dot{R}^4$ is also invariant, as well as the Hamiltonian H, so we may reduce the Hamiltonian system $(T^*\dot{R}^4, \Omega, H)$ with respect to the $U(1)$ action M_s. The result is the following proposition established by Iwai and Uwano (1986).

Proposition. Let $\mu\in R$. Then

$$M^{-1}(\mu)/U(1)=T^*\dot{R}^3. \tag{21}$$

Moreover, the reduction of the form Ω and the Hamiltonian H give Ω_μ and H_μ, i.e. the result of reduction is the MIC-Kepler problem (1).

Remark. By lemma 1 we see that the orbits of the $U(1)$ action K_t are closures of the orbits of the Hamiltonian H, considered as subsets of $\dot{C}^4$. We see also from (9) and (13) that

$$D\subset M^{-1}(0)\subset\dot{C}^4 \tag{22}$$

i.e. all non-closed orbits of the conformal Kepler problem are included in the invariant set $M^{-1}(0)\cap T^*\dot{R}^4$ (we consider, of course, only orbits lying in the negative-energy hypersurfaces of H). Thus we see that for $\mu\neq 0$, the Hamiltonian system $(T^*\dot{R}^3, H_\mu, \Omega_\mu)$ has only closed orbits. For $\mu=0$, we have the standard Kepler problem and some non-closed orbits (collisions with the central body). We regularise the lifted problem (the flow of the conformal Kepler problem on $T^*\dot{R}^4$) if we consider, instead of H, by lemma 1, the Hamiltonian K and extend $T^*\dot{R}^4$ to $\dot{C}^4$. It is easy to see from (9) and (16) that each orbit of the action K_t has at most two common points with the complementary set $D\backslash\{0\}$. In particular, no orbit of the K_t action is contained in $\dot{C}^4\backslash T^*\dot{R}^4=D\backslash\{0\}$, and we have a one-to-one correspondence between orbits of the MIC-Kepler problem on the energy hypersurface $H=E$, $E<0$, and orbits of the torus action $U(1)\times U(1)$ defined by (18) and (19) on $J^{-1}(\mu,4\alpha)$. We have proved the following lemma.

Lemma 3. $\mathscr{C}_\mu(E)=J^{-1}(\mu,4\alpha)/U(1)\times U(1)$.

Proof of theorem 1. Because of lemma 3, we have to prove that $J^{-1}(\mu,4\alpha)/U(1)\times U(1)$ is diffeomorphic to

(i) $P^1\times P^1$ if $\lambda|\mu|<2\alpha$

(ii) P^1 if $\lambda|\mu|=2\alpha$

(iii) ϕ otherwise.

Using (12) and (13) we see that the system, $K=4\alpha$, $M=\mu$, is equivalent to

$$|z_1|^2+|z_2|^2=4(2\alpha+\lambda\mu)\qquad |z_3|^2+|z_4|^2=4(2\alpha-\lambda\mu) \tag{23}$$

whence

$$J^{-1}(\mu, 4\alpha) = \begin{cases} S^3 \times S^3 & \text{when } \lambda|\mu| < 2\alpha \\ S^3 & \text{when } \lambda|\mu| = 2\alpha \\ \phi & \text{when } \lambda|\mu| > 2\alpha. \end{cases} \tag{24}$$

Define a projection (two Hopf maps)

$$p : S^3 \times S^3 \to P^1 \times P^1$$

by

$$p(z_1, z_2, z_3, z_4) = ((z_1, z_2), (z_3, z_4)) \tag{25}$$

where (z_1, z_2), (z_3, z_4) are homogeneous coordinates on $P^1 \times P^1$. By lemma 2, the map p is exactly the factor map

$$J^{-1}(\mu, 4\alpha) \to J^{-1}(\mu, 4\alpha)/U(1) \times U(1).$$

This proves statement (i) of theorem 1. Obviously, the restriction of the projection p to the non-zero factor also gives the factor map needed to prove the statement (ii). The statement (iii) is trivial.

Anyway, the case $\lambda|\mu| = 2\alpha$ deserves some comment. We need to consider the sets

$$D' = \{z \in \dot{C}^4;\ |z_1|^2 + |z_2|^2 = 0\} \qquad D'' = \{z \in \dot{C}^4;\ |z_3|^2 + |z_4|^2 = 0\}. \tag{26}$$

The action of $U(1) \times U(1)$ on C^4 defined by (18) and (19) is free exactly on $\dot{C}^4 \backslash (D' \cup D'')$. On each of the sets D', D'', the actions (18) and (19) coincide up to a change of parameter. Thus the orbits of the MIC-Kepler problem, which are images of K_t orbits contained in D' or D'' under reduction with respect to M_s, shrink to single points. Note that $(D' \cup D'') \cap M^{-1}(0) = \phi$. An easy calculation yields that, for any $\mu \neq 0$ the energy value E, such that $\sqrt{-8E}|\mu| = 2\alpha$, corresponds to a minimum of the potential function. Thus the statement of our theorem for the case $\lambda|\mu| = 2\alpha$ means that, in these conditions, we have a 2-sphere of stationary points. We note also that $D' \cup D''$ are exactly the points of $\dot{C}^4$ where the moment J degenerates.

We have to compute the reduced symplectic form. In non-homogeneous coordinates

$$(\zeta_1, \zeta_2) = (z_2/z_1, z_4/z_3)$$

on $P^1 \times P^1$, the map p is given by

$$p(z_1, z_2, z_3, z_4) = (\zeta_1, \zeta_2). \tag{27}$$

By the reduction theorem we have

$$p^* \Omega_\mu(E) = \Omega_{|S^3 \times S^3}$$

where $S^3 \times S^3$ are exactly the spheres defined in (23). Thus

$$\begin{aligned} p^*\left(\frac{\mathrm{i}(2\alpha + \lambda\mu)}{\lambda} \frac{\mathrm{d}\zeta_1 \wedge \mathrm{d}\bar{\zeta}_1}{(1+|\zeta_1|^2)^2} + \frac{\mathrm{i}(2\alpha - \lambda\mu)}{\lambda} \frac{\mathrm{d}\zeta_2 \wedge \mathrm{d}\bar{\zeta}_2}{(1+|\zeta_2|^2)^2}\right) \\ = \frac{\mathrm{i}}{\lambda}\left((2\alpha + \lambda\mu)\frac{\mathrm{d}z_1 \wedge \mathrm{d}\bar{z}_1 + \mathrm{d}z_2 \wedge \mathrm{d}\bar{z}_2}{|z_1|^2 + |z_2|^2} + (2\alpha - \lambda\mu)\frac{\mathrm{d}z_3 \wedge \mathrm{d}\bar{z}_3 + \mathrm{d}z_4 \wedge \mathrm{d}\bar{z}_4}{|z_3|^2 + |z_4|^2}\right) \\ = \Omega_{|S^3 \times S^3} \end{aligned}$$

because of (10) and (23). Theorem 1 is proved.

We recall here only the relevant facts from the geometric quantisation scheme for a compact Kaehler manifold (X, ω) with a polarisation given by the complex structure. The quantum conditions on the form ω amount to

$$(1/2\pi)[\omega]-\tfrac{1}{2}c_1(X)\in H^2(X, Z). \tag{28}$$

The quantum states of the system are identified with the holomorphic sections of a line bundle L over X such that

$$c_1(L)=(1/2\pi)[\omega]-\tfrac{1}{2}c_1(X). \tag{29}$$

Moreover, in order that the space of quantum states $H^0(X, L)$ is not empty, we must have

$$\int_\sigma c_1(L)\geqslant 0 \qquad \text{for all } \sigma\in H_2(X, Z). \tag{30}$$

Proof of theorem 2. Let $E<0$. Then in order that E be a quantum level, the corresponding form $\Omega_\mu(E)$ (see (4)) must satisfy conditions (28)-(30). We have (Griffiths and Harris 1978):

$$c_1(P^1\times P^1)=2(\omega_1+\omega_2) \tag{31}$$

where ω_1, ω_2 are defined in (5) and $[\omega_1]$, $[\omega_2]$ generate

$$H^2(P^1\times P^1, \mathbb{Z})=\mathbb{Z}\oplus\mathbb{Z}.$$

Equations (28)-(31) yield

$$(1/2\pi)\Omega_\mu(E)=N_1\omega_1+N_2\omega_2 \tag{32}$$

for some integers N_1, $N_2\geqslant 1$. We combine (4) and (32) to obtain

$$2\alpha+\lambda\mu=\lambda N_1$$

$$2\alpha-\lambda\mu=\lambda N_2$$

whence

$$\mu=\tfrac{1}{2}(N_1-N_2) \qquad \lambda=4\alpha/(N_1+N_2). \tag{33}$$

It is (at least mathematically) sensible to prequantise also the orbit manifolds when $\lambda|\mu|=2\alpha$. It is curious to remark that the procedure gives exactly the quantisation of the magnetic charge μ which was obtained independently by the above argument (see (33)) and also may be obtained if we apply, as did Ryder (1980) and Crampin (1981), prequantisation to the symplectic form Ω_μ of the original MIC-Kepler phase space. Note that the singular momentum levels do not occur at all for the standard Kepler problem $\mu=0$. Let us take, e.g., $\mu>0$, $\lambda\mu=2\alpha$. Then $H_\mu^{-1}(E)/U(1)\simeq P^1$ and we have $H^2(P^1)=\mathbb{Z}[\omega_1]$, $c_1(P^1)=2[\omega_1]$ whence (4), (28)-(30) give $\mu=\tfrac{1}{2}\mathcal{N}$ for some positive integer $\mathcal{N}\geqslant 1$.

Introducing a new non-negative (half)-integral variable $N=\tfrac{1}{2}(N_1+N_2)$, we obtain $N_1=N+\mu$, $N_2=N-\mu$, where $N\geqslant|\mu|+1$, and finally obtain the energy spectrum of the MIC-Kepler problem in the form

$$E_N=-\alpha^2/2N^2 \qquad N=|\mu|+1, |\mu|+2, \ldots. \tag{34}$$

The multiplicities of these energy levels, $m(E_N)$, coincide with the dimensions of the spaces of holomorphic sections of the line bundles L_N over $P^1 \times P^1$. If $L_N \rightarrow P^1 \times P^1$ is such a quantum bundle and

$$c_1(L_N) = (N_1 - 1)[\omega_1] + (N_2 - 1)[\omega_2] \tag{35}$$

then, by the Riemann-Roch-Hirzebruch theorem for compact complex surfaces (Hirzebruch 1966) and the Kodaira vanishing theorem (Griffiths and Harris 1978) we have

$$m(E_N) = \dim H^0(P^1 \times P^1, L_N) = N_1 N_2 = N^2 - \mu^2. \tag{36}$$

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